8.1) a) there are at least 7 cycles of length 4; the´1´s involved in these cycles are seen in matrix $\boldsymbol{H}$ below:

$$\boldsymbol{H}=\begin{bmatrix} 0 & \mathbf{1} & 0 & 1 & 0 & 1 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 1 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{1} & 0 \\ 0 & 0 & 1 & 0 & 1 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 1 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

b) In order to maintain $s=3$, ´1´s should be moved along the columns by replacement of ´0´s by ´1´s and vice versa, in the same column. Every position of a ´0´ in the above matrix can not be filled by replacement with a ´1´ unless another cycle is formed.

c) In general, it will correspond to another code.

8.2) a) The rank of the matrix (that is, the number of linearly independent columns or rows of the matrix) is 4. The addition (in the binary field) of columns 1, 2, 3 and 6 is the all zero vector, thus the rank of this matrix is 4.

The cyclic parity check matrix is $\boldsymbol{H}=\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$.

The parity check matrix in systematic form is obtained by performing row operations over the above matrix. Thus, the following operations:

$$row1+row2+row3 \Rightarrow row1$$
$$row2+row3+row4 \Rightarrow row2$$
$$row3+row4 \Rightarrow row3$$

generates the matrix in systematic form:

$$\boldsymbol{H}_{sys} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

The corresponding generator matrix is obtained taking into account that when the form of the parity check matrix is $\boldsymbol{H}_{sys} = [\boldsymbol{I}_{n-k} \quad \boldsymbol{P}^T]$, the generator matrix of the block code is $\boldsymbol{G}_{sys} = [\boldsymbol{P} \quad \boldsymbol{I}_k]$. Thus:

$$\boldsymbol{G}_{sys} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

b) The average number of 1's per row is $v = 3$, and the average number of 1's per column is $s = 12/7$ in the cyclic PC matrix. The average number of 1's per row is $v = 13/4$, and the average number of of 1's per column is $s = 13/7$ in the systematic PC matrix. The code rate for the cyclic PC code is:

$$\frac{k}{n} = \frac{v-s}{v} = \frac{3-12/7}{3} = \frac{9}{3x7} = \frac{3}{7}$$

The code rate for the systematic PC code is:

$$\frac{k}{n} = \frac{v-s}{v} = \frac{13/4-13/7}{13/4} = \frac{(91-52)x4}{13x28} = \frac{3}{7}$$

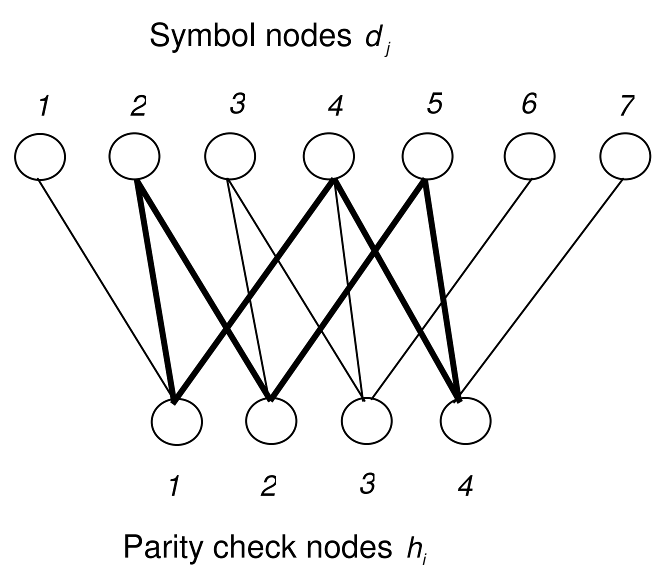
The code rate is equal to $k/n = 3/7$, obtained for instance from the size of the corresponding generator matrix.

The table of non-zero codewords and their corresponding weights for this code is the following:

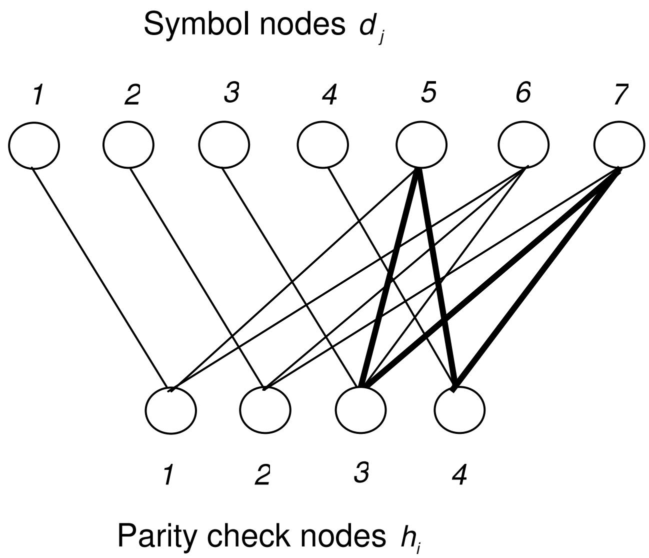
								w
0	1	1	1	0	0	1		4
1	1	1	0	0	1	0		4
1	0	0	1	0	1	1		4
1	0	1	1	1	0	0		4
1	1	0	0	1	0	1		4
0	1	0	1	1	1	0		4
0	0	1	0	1	1	1		4

Since the code is linear, the minimum weight is also the minimum distance of the code, thus, $d_{min} = 4$.

c) The Tanner graph for the cyclic PC code is seen in the following figure. A cycle of the shortest length, 6, is seen in bold line. There are no cycles of length 4.



The Tanner graph for the systematic PC code is seen in the following figure. A cycle of the shortest length, 4, is seen in bold lines.



The best option for a LDPC code decoded using the SPA is the cyclic PC matrix, since the shortest length of a given cycle (in this case 6) is longer than for the systematic PC matrix (shortest cycle of length 4) .

8.3) a) The code vector for the message vector $\boldsymbol{m} = (100)$ is the first row of the generator matrix G_{sys}, that is, $\boldsymbol{c} = (1011100)$.

b) The following is a detail of calculations for the cyclic graph, for the two first iterations. The first step is to determine coefficients f_j^0 and f_j^1. The following table shows these values:

	1	2	3	4	5	6	7
f_j^0	0.0358	0.4059	0.0013	0.0050	0.4055	0.4256	0.3151
f_j^1	0.4432	0.0873	0.2181	0.3292	0.0134	0.0679	0.1673

These coefficients allow us to determine the values of Q_{ij}^0 and Q_{ij}^1 in the initialization:

Q_{ij}^0	1	2	3	4	5	6	7
1	0.0358	0.4059	0	0.0050	0	0	0
2	0	0.4059	0.0013	0	0.4055	0	0
3	0	0	0.0013	0.0050	0	0.4256	0
4	0	0	0	0.0050	0.4055	0	0.3151

Q_{ij}^1	1	2	3	4	5	6	7
1	0.4432	0.0873	0	0.3292	0	0	0
2	0	0.0873	0.2181	0	0.0134	0	0
3	0	0	0.2181	0.3292	0	0.0679	0
4	0	0	0	0.3292	0.0134	0	0.1673

Using the Mackay-Neal modified SP algorithm, we can also determine coefficients δR_{ij} .

Here starts the iterative process of decoding:

δR_{ij}	1	2	3	4	5	6	7
1	-0.1033	0.1321	0	-0.1298	0	0	0
2	0	-0.0850	0.1249	0	-0.0691	0	0
3	0	0	-0.1160	-0.0775	0	0.0703	0
4	0	0	0	0.0580	-0.0479	0	-0.1271

Values of R_{ij}^{0} and R_{ij}^{1} in the first iteration are now determined:

R_{ij}^{0}	1	2	3	4	5	6	7
1	0.4483	0.5660	0	0.4351	0	0	0
2	0	0.4575	0.5625	0	0.4655	0	0
3	0	0	0.4420	0.4612	0	0.5351	0
4	0	0	0	0.5290	0.4760	0	0.4364

R_{ij}^{1}	1	2	3	4	5	6	7
1	0.5517	0.4340	0	0.5649	0	0	0
2	0	0.5425	0.4375	0	0.5345	0	0
3	0	0	0.5580	0.5388	0	0.4649	0
4	0	0	0	0.4710	0.5240	0	0.5636

And then values of Q_{ij}^{0} and Q_{ij}^{1} can be updated:

Q_{ij}^0	1	2	3	4	5	6	7
1	0.0748	0.7968	0	0.0145	0	0	0
2	0	0.8585	0.0047	0	0.9648	0	0
3	0	0	0.0077	0.0130	0	0.8623	0
4	0	0	0	0.0100	0.9633	0	0.6533

Q_{ij}^1	1	2	3	4	5	6	7
1	0.9252	0.2032	0	0.9855	0	0	0
2	0	0.1415	0.9953	0	0.0352	0	0
3	0	0	0.9923	0.9870	0	0.1377	0
4	0	0	0	0.9900	0.0367	0	0.3467

a decision can be taken in the first iteration, by calculating coefficients of the estimates Q_j^0 and Q_j^1 :

	1	2	3	4	5	6	7
Q_j^0	0.0161	0.1051	0.0003	0.0005	0.0899	0.2277	0.1375
Q_j^1	0.2445	0.0205	0.0532	0.0472	0.0038	0.0316	0.0943

The decoded vector is then:

$$d_{1rst\ iter} = (1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0)$$

This estimation results into a non zero syndrome, and iterations will continue.

The decoded message in this first iteration is $m_{1rst\ iter} = (0 \quad 0 \quad 0)$

The second iteration starts with the calculation of the updated values of δR_{ij} :

δR_{ij}	1	2	3	4	5	6	7
1	-0.5765	0.8258	0	-0.5049	0	0	0
2	0	-0.9208	0.6665	0	-0.7102	0	0
3	0	0	-0.7058	-0.7136	0	0.9590	0
4	0	0	0	0.2840	-0.3004	0	-0.9082

Values of R_{ij}^{0} and R_{ij}^{1} in the second iteration are now determined:

R_{ij}^{0}	1	2	3	4	5	6	7
1	0.2117	0.9129	0	0.2476	0	0	0
2	0	0.0396	0.8332	0	0.1449	0	0
3	0	0	0.1471	0.1432	0	0.9795	0
4	0	0	0	0.6420	0.3498	0	0.0459

R^1_{ij}	1	2	3	4	5	6	7
1	0.7883	0.0871	0	0.7524	0	0	0
2	0	0.9604	0.1668	0	0.8551	0	0
3	0	0	0.8529	0.8568	0	0.0205	0
4	0	0	0	0.3580	0.6502	0	0.9541

And then values of Q^0_{ij} and Q^1_{ij} can be updated:

Q^0_{ij}	1	2	3	4	5	6	7
1	0.0748	0.1609	0	0.0046	0	0	0
2	0	0.9799	0.0010	0	0.9420	0	0
3	0	0	0.0291	0.0089	0	0.8623	0
4	0	0	0	0.0008	0.8364	0	0.6533

Q^1_{ij}	1	2	3	4	5	6	7
1	0.9252	0.8391	0	0.9954	0	0	0
2	0	0.0201	0.9990	0	0.0580	0	0
3	0	0	0.9709	0.9911	0	0.1377	0
4	0	0	0	0.9992	0.1636	0	0.3467

a decision can be taken in the second iteration, by calculating coefficients of the estimates Q_j^0 and Q_j^1 :

	1	2	3	4	5	6	7
Q_j^0	0.0076	0.0147	0.0002	0.0001	0.0206	0.4169	0.0145
Q_j^1	0.3493	0.0073	0.0310	0.0760	0.0075	0.0014	0.1596

The decoded vector is then:

$$d_{2nd\ iter} = (1\quad 0\quad 1\quad 1\quad 0\quad 0\quad 1)$$

This estimation results into a non zero syndrome, and iterations will continue.

The decoded message in this second iteration is $m_{2nd\ iter} = (0\quad 0\quad 1)$.

The above described calculations can be also performed for the following iterations. The decoded vectors in the following iterations are:

Third iteration: $d_{3rd\ iter} = (1011101)$, decoded message $m_{3rd\ iter} = (1\quad 0\quad 1)$

Fourth iteration: $d_{4th\ iter} = (1011100)$, decoded message $m_{4th\ iter} = (1\quad 0\quad 0)$, successful decoding, syndrome is zero.

In the case of the systematic graph, the decoded vectors in successive iterations are:

first iteration: $d_{1rst\ iter} = (1011000)$,

second iteration $d_{2nd\ iter} = (1011011)$,

third iteration: $d_{3rd\ iter} = (1011011)$,

fourth iteration: $d_{4th\ iter} = (1011000)$,

fifth iteration: $d_{5th\ iter} = (10110101)$,

sixth iteration: $d_{6th\ iter} = (1011101)$, unsuccessful decoding in six iterations.

8.4) a) The decoder can determine after a little number of iterations the output vector, decoded bit by bit. The solution provided by the decoder is not a codeword, and there are also convergence problems, since the iterative decoding provides always the same decoded vector after a given number of solutions. It would reach the maximum number of iterations giving this solution at the end of the decoding. After a little number of iterations, the estimates for each bit are the following:

	1	2	3	4	5
Q_j^0	0.1019	0.4091	0.0990	0.4091	0.1855
Q_j^1	0.0990	0.0091	0.1019	0.0091	0.0046

The decoded vector is $d = (0 \quad 0 \quad 1 \quad 0 \quad 0)$, which is not a code vector. Estimates for bits 1 and 3 are however quite close to each other, and slight differences in their values can make the decoded vector be one of two codewords that are $c_1 = (0 \quad 0 \quad 0 \quad 0 \quad 0)$ (the transmitted codeword) and $c_2 = (1 \quad 0 \quad 1 \quad 0 \quad 0)$. Estimates for bits at positions 2, 4 and 5 are however very well defined.

This performance can be seen by using the program codec_ldpc_P_8_4_ch8.m, which can be downloaded form the website of the book.

b) The decoder can not be efficient for large number of iterations because of the characteristics of the code. Connections between symbol nodes and parity check nodes are not enough for effective belief propagation.