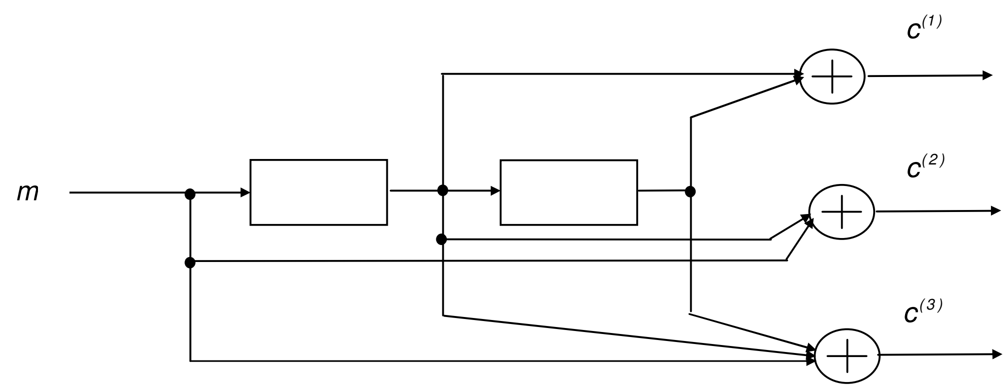
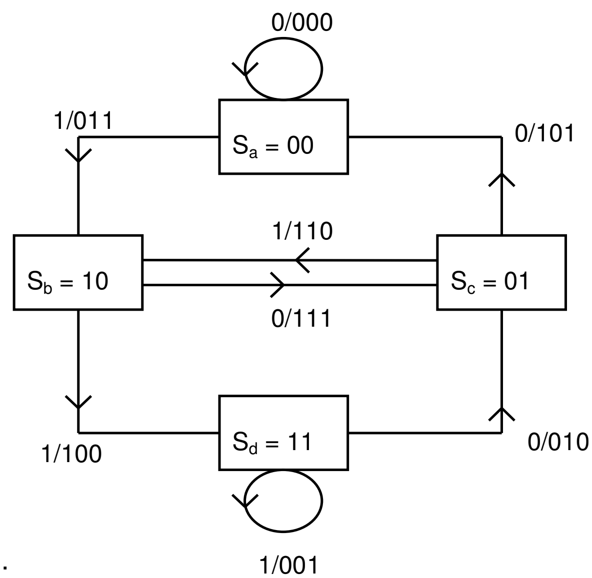
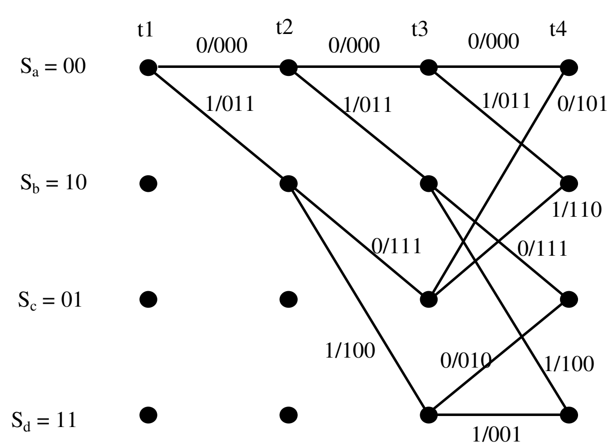
6.1)
The following figure corresponds to the convolutional encoder with generator polynomials $g^{(1)}(D)=D+D^2$, $g^{(2)}(D)=1+D$ and $g^{(3)}(D)=1+D+D^2$.



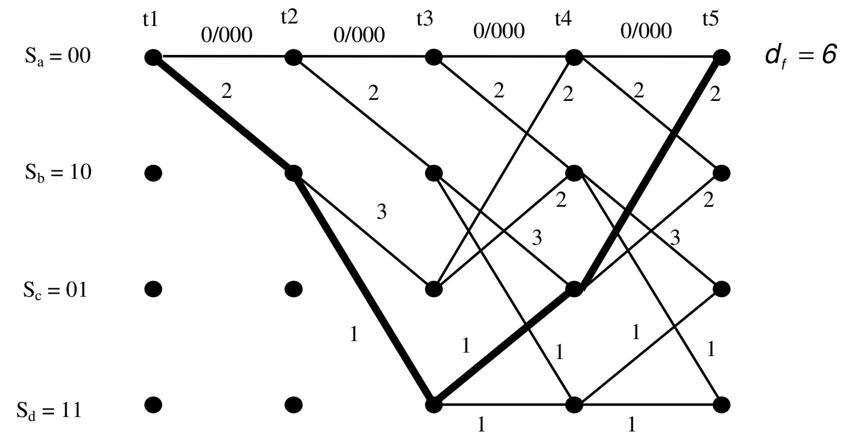
The corresponding state diagram is the following:



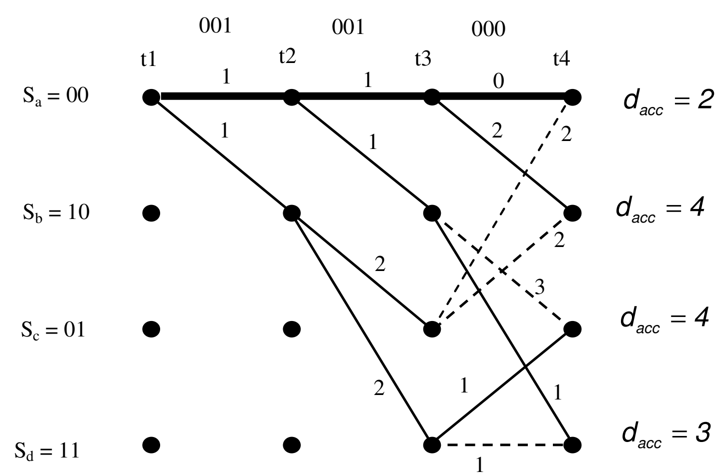
And the trellis is:



The following figure shows the path that determines the minimum Hamming free distance of this code, which is equal to $d_f = 6$. Transitions of the trellis are denoted by its Hamming weight with respect to the all-zero sequence.



For the message sequence $\boldsymbol{m} = (000)$ the code sequence is $\boldsymbol{c} = (000 \quad 000 \quad 000)$. If the received sequence is $\boldsymbol{s}_r = (001 \quad 001 \quad 000)$, then the decoding performs as follows:



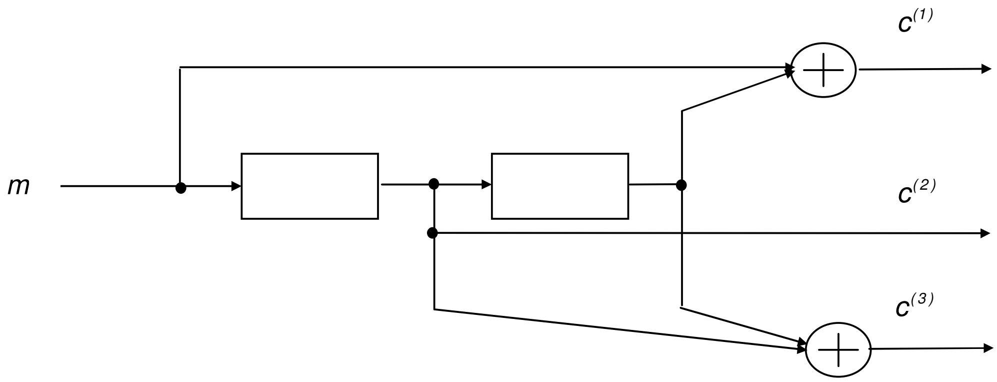
At the time instant t_4 a decision is taken. The discarded paths are in doted lines. Transitions of the trellis are denoted by its Hamming weight with respect to the received sequence. The minimum accumulated Hamming distance at that point is $d_{acc} = 2$, and the decoded path is the code sequence $\boldsymbol{c} = (000 \quad 000 \quad 000)$. Thus, the decoding was successful, and two errors were corrected.

The code is not catastrophic because there are no loops of zero weight at states different form the all-zero state $S_a = (00)$.

Another way of determining this condition is by verifying if the generator polynomials have some common factor different from D^l.

6.2)

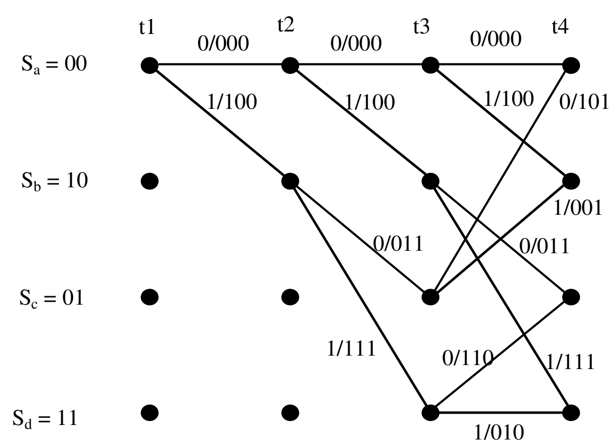
$k=1,\ n=3,\ K=2,\ g^{(1)}(D)=1+D^2,\ g^{(2)}(D)=D$ and $g^{(3)}(D)=D+D^2$



The following table describes the transitions of the corresponding trellis.

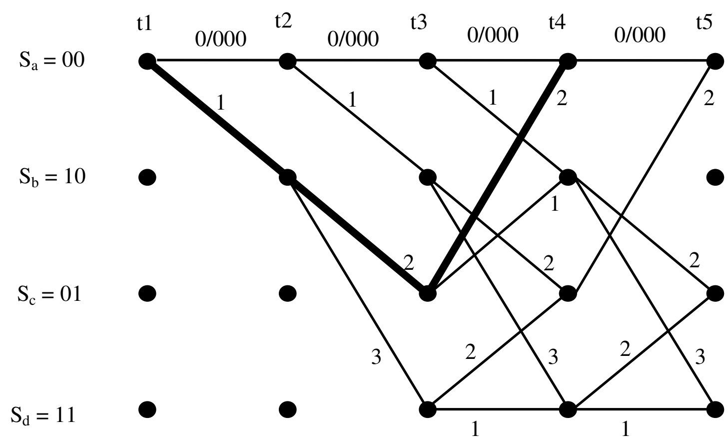
Input m_i	State at t_i	State at t_{i+1}	$c^{(1)}$	$c^{(2)}$	$c^{(3)}$
-	0 0	0 0	-	-	-
0	0 0	0 0	0	0	0
1	0 0	1 0	1	0	0
0	0 1	0 0	1	0	1
1	0 1	1 0	0	0	1
0	1 0	0 1	0	1	1
1	1 0	1 1	1	1	1
0	1 1	0 1	1	1	0
1	1 1	1 1	0	1	0

The trellis is the following:



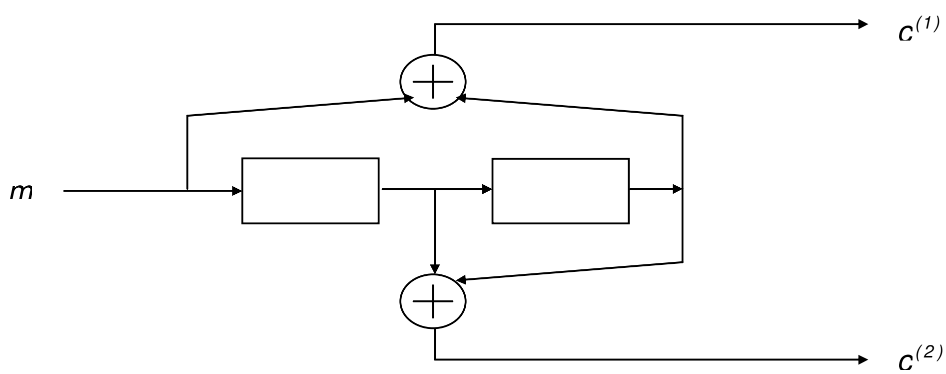
The minimum Hamming free distance is calculated considering the path that emerges from and arrives at the all-zero state that has the minimum Hamming weight.

$d_f = 5$



The code is systematic, because a delayed version of the message sequence is equal to the sequence generated at output $c^{(2)}$.

6.3) For the convolutional encoder seen in the following figure, the generator polynomials are:



$g^{(1)}(D) = 1 + D^2$

$g^{(2)}(D) = D + D^2$

Since:

$g^{(1)}(D) = 1 + D^2 = (1 + D)(1 + D)$

and

$g^{(2)}(D) = D + D^2 = D(1 + D)$,

then the MCD is $MCD = 1 + D \neq D^l$, and the code is catastrophic.

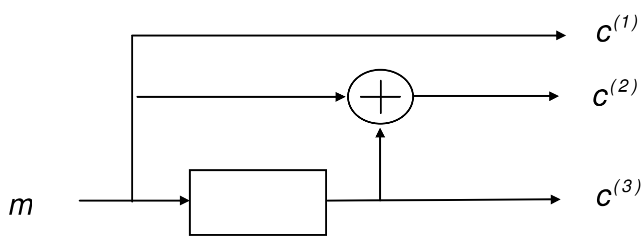
The following table describes the transitions of the corresponding trellis.

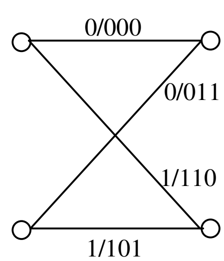
Input m_i	State at t_i	State at t_{i+1}	$c^{(1)}$	$c^{(2)}$
-	0 0	0 0	-	-
0	0 0	0 0	0	0
1	0 0	1 0	1	0
0	0 1	0 0	1	1
1	0 1	1 0	0	1
0	1 0	0 1	0	1
1	1 0	1 1	1	0
0	1 1	0 1	1	0
1	1 1	1 1	**0**	**0**

The loop at state $S_d(11)$ has zero weight, and the code is catastrophic.

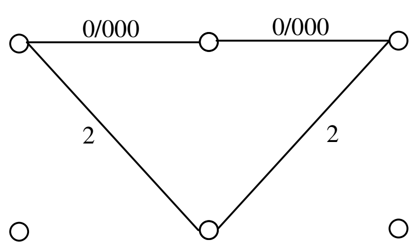
For the input sequence $\boldsymbol{m} = (101)$, the output sequence is $\boldsymbol{c} = (10 \quad 01 \quad 01)$
(Please see in answers to problems in the book that there is a typing error for this solution).

6.4) The convolutional encoder seen in the following figure has the following trellis:

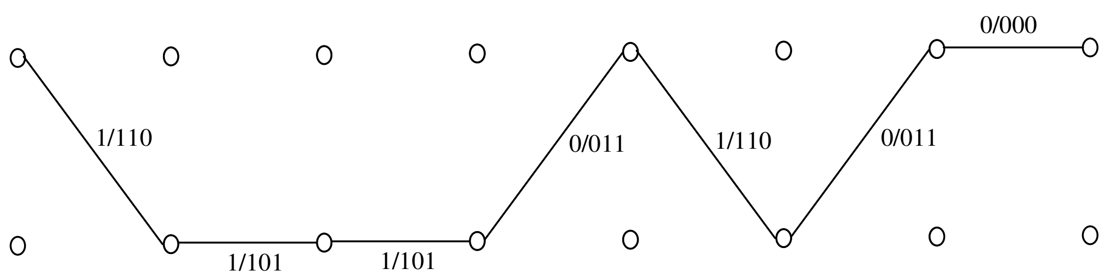




The constraint length is equal to $K + 1 = 2$
and the minimum Hamming distance is equal to $d_f = 4$ and it is evaluated as seen in the following figure:

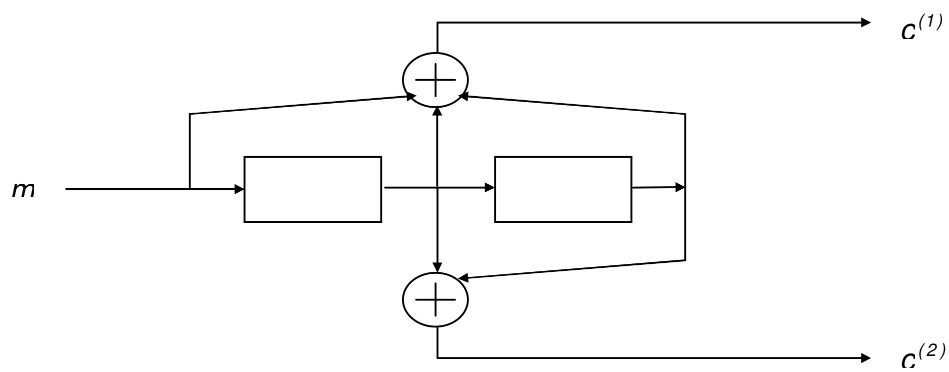


The output sequence for the input sequence $\boldsymbol{m} = (1110100)$ is $\boldsymbol{c} = (110 \;\; 101 \;\; 101 \;\; 011 \;\; 110 \;\; 011 \;\; 000)$

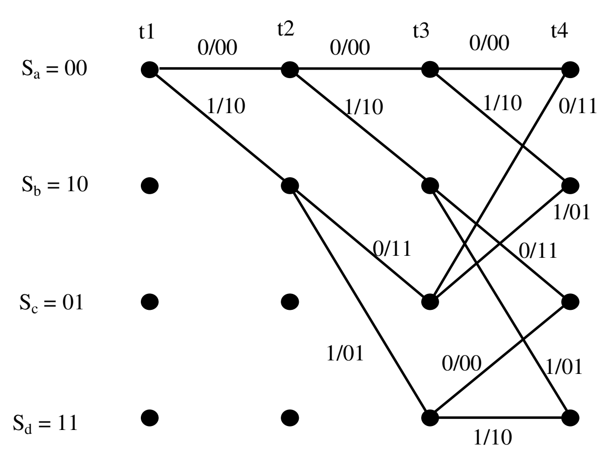


6.5)

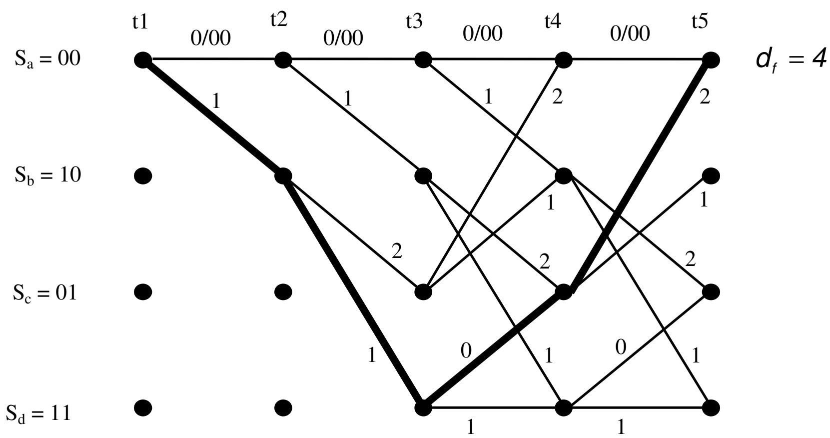
For the convolutional encoder seen in the following figure:



The trellis is:



And the minimum Hamming distance is equal to $d_f = 4$:



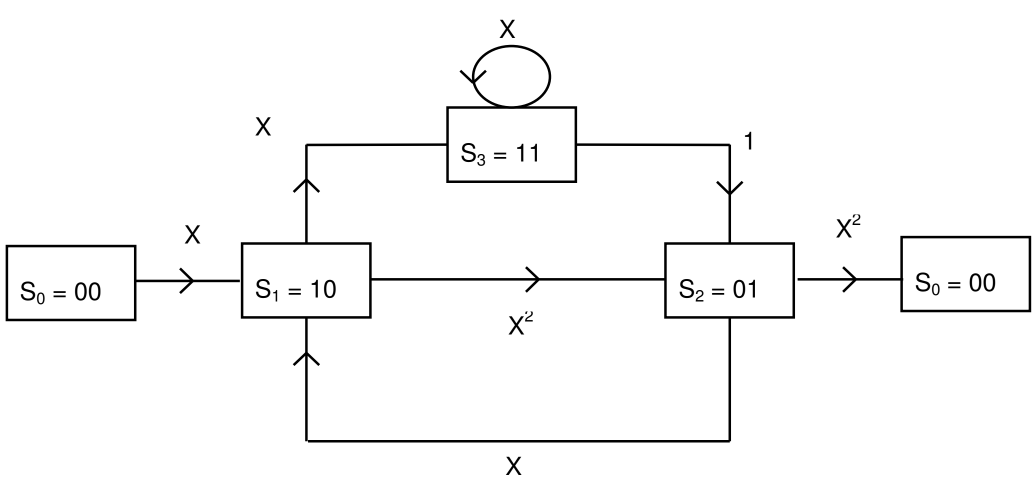
The impulse response is the output for the input sequence $\boldsymbol{m} = (10000\ldots)$. This can be determined in different ways. One of them is by means of the trellis representation. Another way is obtained by inspection of the generator polynomials, which are essentially defined as the impulse responses of the FSSM:

$$g^{(1)}(D) = 1 + D + D^2$$
$$g^{(2)}(D) = D + D^2$$

if $m(D) = 1$, then:

$$c_{IR} = (10 \quad 11 \quad 11)$$

The generating function is obtained using the modified state diagram seen in the following figure:



$$S_0 = 1, \quad S_1 = X + XS_2$$

$S_2 = X^2 S_1 + S_3$,
$S_3 = XS_1 + XS_3$,
$T(X) = X^2 S_2$

$$S_3 = S_1 \frac{X}{1-X}, \qquad S_2 = S_1\left[X^2 + \frac{X}{1-X}\right] = S_1\left[\frac{X^2 - X^3 + X}{1-X}\right],$$

$$S_1 = \frac{1-X}{X+X^2-X^3}S_2, \quad \frac{1-X}{X+X^2-X^3}S_2 = X + XS_2,$$

$$S_2 - XS_2 = X^2 + X^3 - X^4 + X^2S_2 + X^3S_2 - X^4S_2,$$

$$\left(1 - X - X^2 - X^3 + X^4\right)S_2 = X^2 + X^3 - X^4,$$

$$S_2 = \frac{X^2 + X^3 - X^4}{1 - X - X^2 - X^3 + X^4},$$

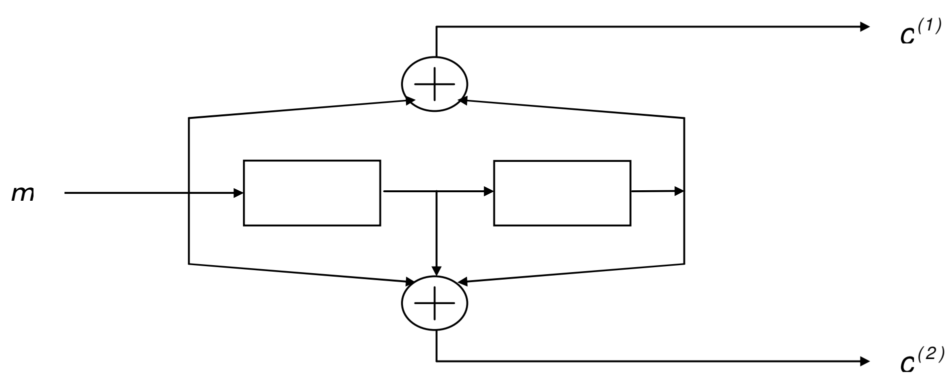
$$T(X) = X^2 S_2 = X^2 \frac{X^2 + X^3 - X^4}{1 - X - X^2 - X^3 + X^4} = \frac{X^4 + X^5 - X^6}{1 - X - X^2 - X^3 + X^4}$$

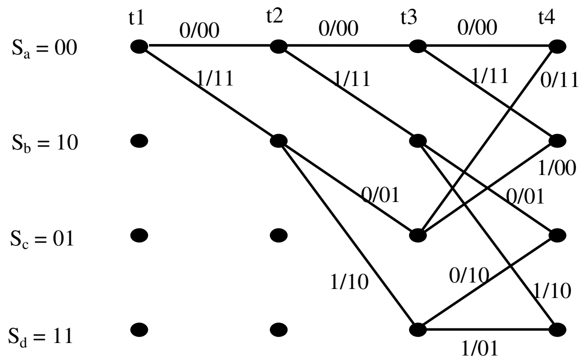
$X^4 + X^5 - X^6 \qquad /1 - X - X^2 - X^3 + X^4$
$-(X^4 - X^5 - X^6 - X^7 + X^8) \qquad X^4 + 2X^5 + 2X^6 + ...$
$-----------------$
$2X^5 + X^7 - X^8$
$-(2X^5 - 2X^6 - 2X^7 - 2X^8 + 2X^9)$
$-----------------$
$2X^6 + 3X^7 + X^8 - 2X^9 ....$

The minimum Hamming free distance is therefore $d_f = 4$.

Since $p = 10^{-3}$ and:
$P_e < A_{df} 2^{d_f} p^{d_f/2} = 1x2^4 x(10^{-3})^2 = 16x10^{-6}$

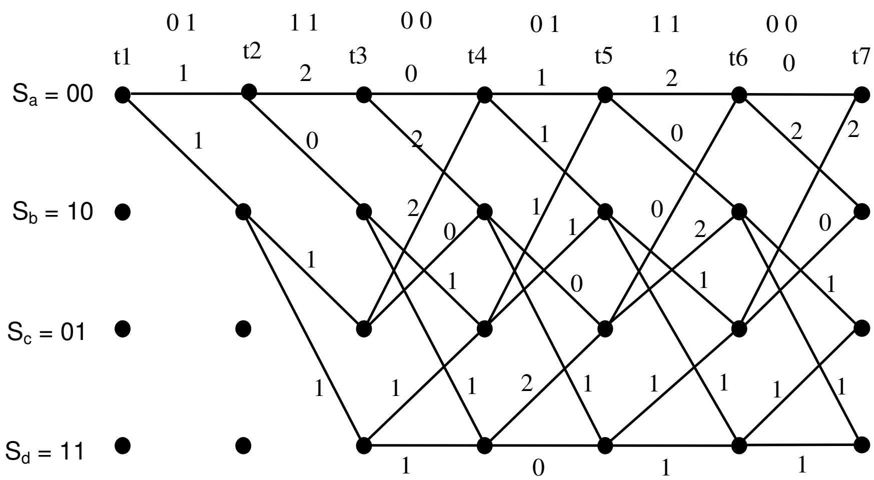
6.6) The convolutional encoder seen in the following figure has a trellis of the form:



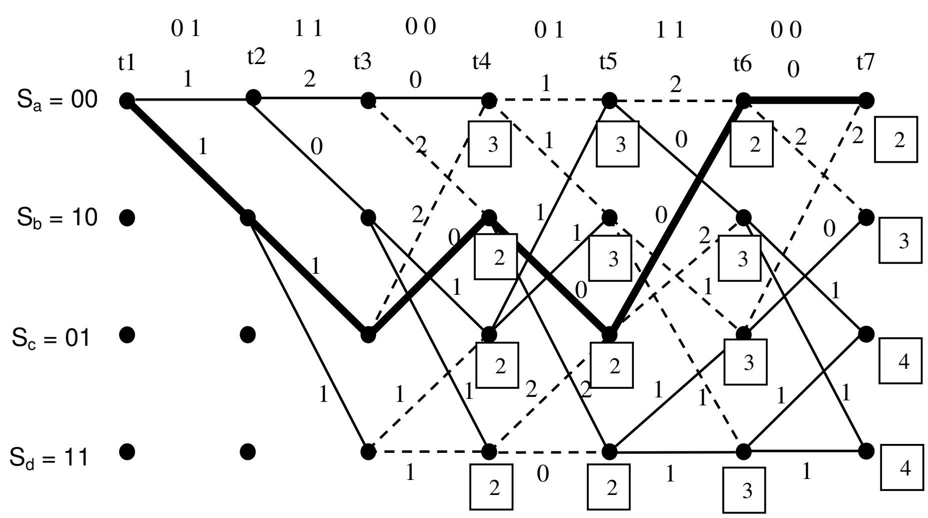


For the received sequence $\boldsymbol{s_r} = (01\ 11\ 00\ 01\ 11\ 00\ 00...)$, the decoding proceeds as follows:

Hamming distances from the received sequence to the transitions outputs for each transition, are seen in the following figure:



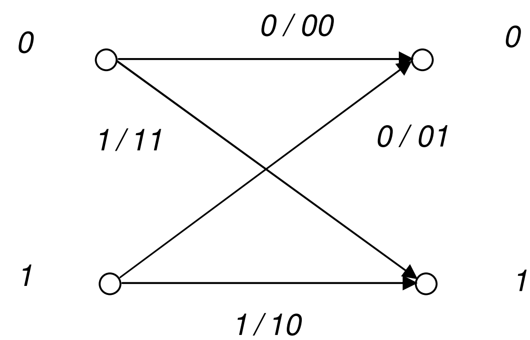
The decoding at the different time instants is shown in the following figure and the accumulative minimum Hamming distances are determined in each step:



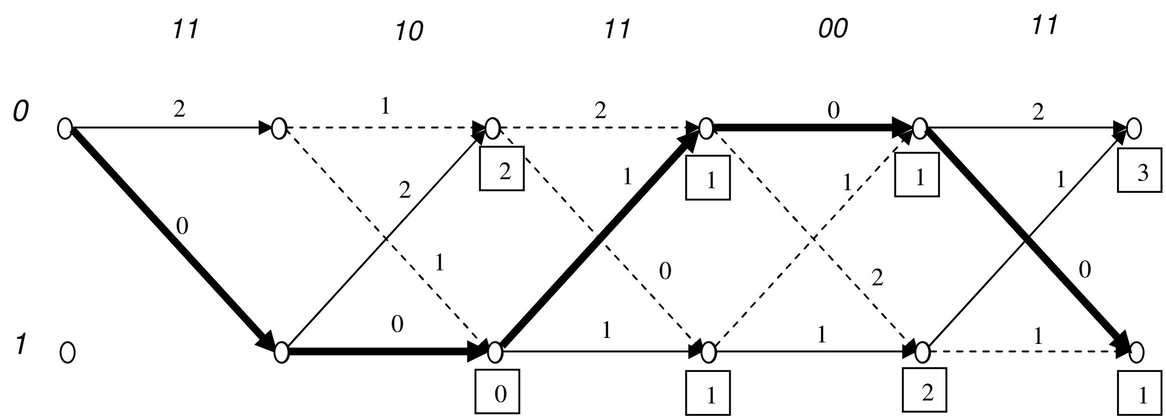
The decoded sequence is then $\boldsymbol{d} = (11\ 01\ 00\ 01\ 11\ 00\ ...)$, and the corresponding message sequence is $\boldsymbol{m} = (1\ 0\ 1\ 0\ 0\ 0\ 0...)$. The decoding was able to correct two errors in the received sequence.

6.7)

The trellis diagram of a binary error correcting convolutional encoder is:

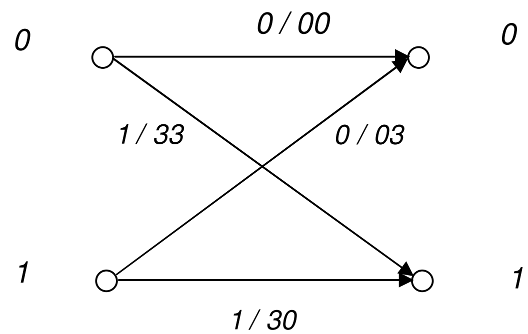


The received sequence is:
$\boldsymbol{s}_r = (11\ 10\ 11\ 00\ 11...)$.
The following figure shows the decoding of this received sequence:

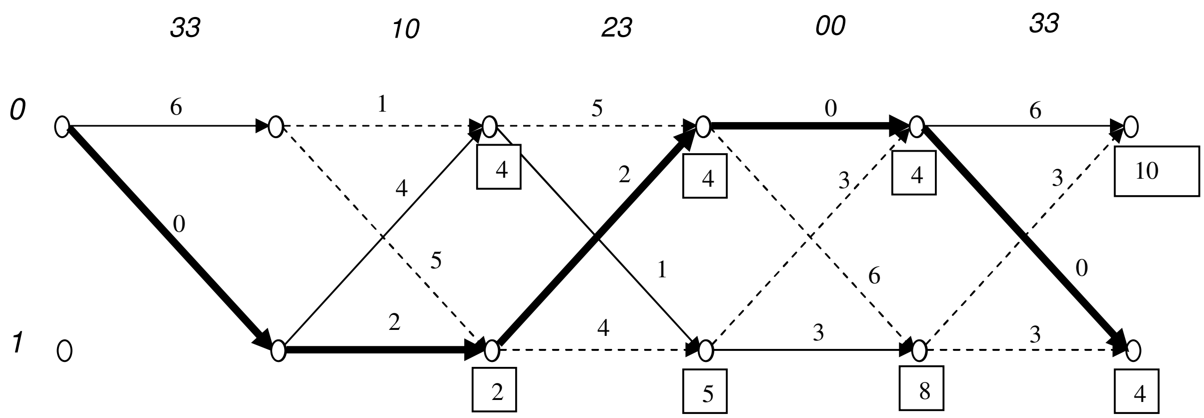


The decoded sequence is then:
$\boldsymbol{d} = (11\ 10\ 01\ 00\ 11...)$ and an error present at the fourth bit of the received sequence is corrected. The message sequence is $\boldsymbol{m} = (1\ 1\ 0\ 0\ 1)$.

A sequence from the encoder of item a) is transmitted over a AWGN channel and is received as the sequence $\boldsymbol{s}_r = (33\ 10\ 23\ 00\ 33)$ after soft decision detection with four levels.
The trellis can be described in terms of the output of the soft decision detection with four levels, and it looks as in the following figure:



The following figure shows the decoding of the received sequence:



The decoded sequence is $\mathbf{d} = (33\ \ 30\ 03\ 00\ \ 33)$ and the error sequence is $\mathbf{e} = (00\ \ 20\ 20\ 00\ \ 00)$.

6.8)

For the convolutional encoder of IIR structure, the transfer function is determined by stating equations in the time domain, and then transform them into the D domain.
In this FSSM:

$$s_0(k) = m(k) + f_2 s_0(k-2) + f_1 s_0(k-1)$$
$$c^{(2)}(k) = a_0 s_0(k) + a_1 s_0(k-1) + a_2 s_0(k-2)$$

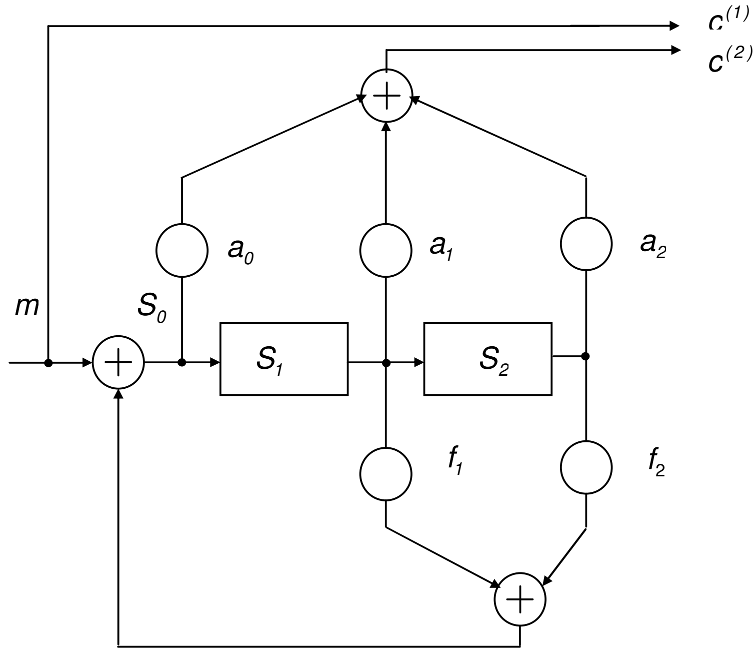
$$s_1(k) = s_0(k-1)$$
$$s_2(k) = s_0(k-2)$$

In the D domain:

$$S_1(D) = DS_0(D)$$
$$S_2(D) = D^2 S_0(D)$$
$$S_0(D) = M(D) + f_1 S_1(D) + f_2 S_2(D) = M(D) + f_1 DS_0(D) + f_2 D^2 S_0(D)$$
$$S_0(D) + f_1 DS_0(D) + f_2 D^2 S_0(D) = M(D)$$

$$S_0(D)=\frac{M(D)}{1+f_1D+f_2D^2}$$

$$C^{(2)}(D)=a_0S_0(D)+a_1DS_0(D)+a_2D^2S_0(D)=(a_0+a_1D+a_2D^2)S_0(D)=(1+D+D^2)\frac{M(D)}{1+f_1D+f_2D^2}$$



The transfer function is:

$$G^{(2)}(D)=\frac{C^{(2)}(D)}{M(D)}=\frac{a_0+a_1D+a_2D^2}{1+f_1D+f_2D^2}$$

Since the code is systematic:

$$\mathbf{G}(D)=\begin{bmatrix}1 & \dfrac{a_0+a_1D+a_2D^2}{1+f_1D+f_2D^2}\end{bmatrix}$$

The state transfer function is determined by obtaining the transfer functions:

$$\frac{S_0(D)}{M(D)}=\frac{1}{1+f_1D+f_2D^2},\ \frac{S_1(D)}{M(D)}=\frac{D}{1+f_1D+f_2D^2},\ \frac{S_2(D)}{M(D)}=\frac{D^2}{1+f_1D+f_2D^2}$$

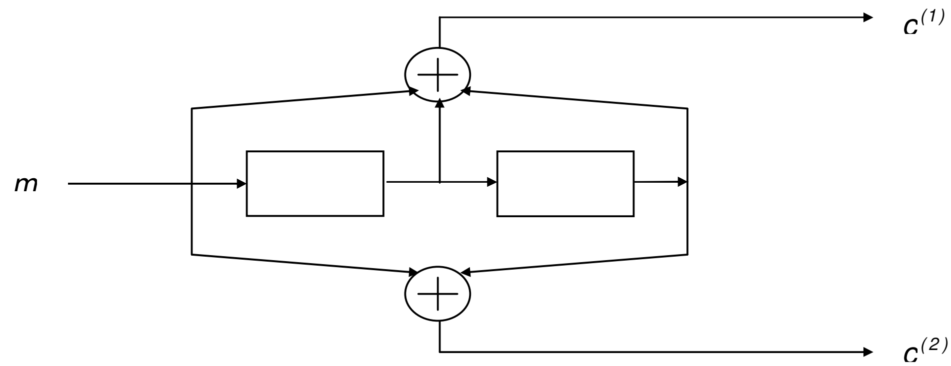
Then:

$$\mathbf{S}(D)=\begin{bmatrix}\dfrac{1}{1+f_1D+f_2D^2} & \dfrac{D}{1+f_1D+f_2D^2} & \dfrac{D^2}{1+f_1D+f_2D^2}\end{bmatrix}$$

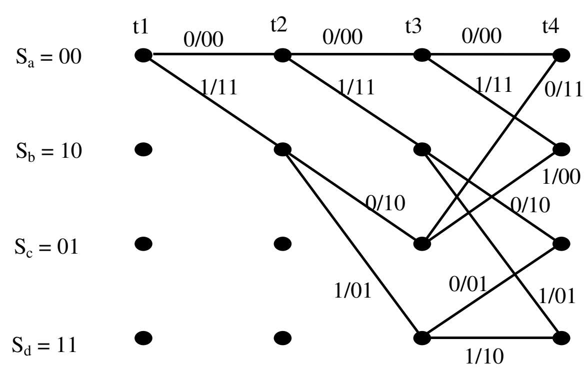
6.9) In this problem there is a typing mistake that results into a difference in the solution of this, and also of the following problem. Solution of the problems 6.9) and 6.10) in the book correspond to a binary convolutional error correcting code with $k=1$, $n=2$,

$K = 2$, $g^{(1)}(D) = 1 + D + D^2$, and $g^{(2)}(D) = 1 + D^2$. The problem statement says that $g^{(2)}(D) = D + D^2$. We will however solve these two cases as an exercise. On the other hand, this problem is the same as the problem 6.5) when $g^{(2)}(D) = D + D^2$.

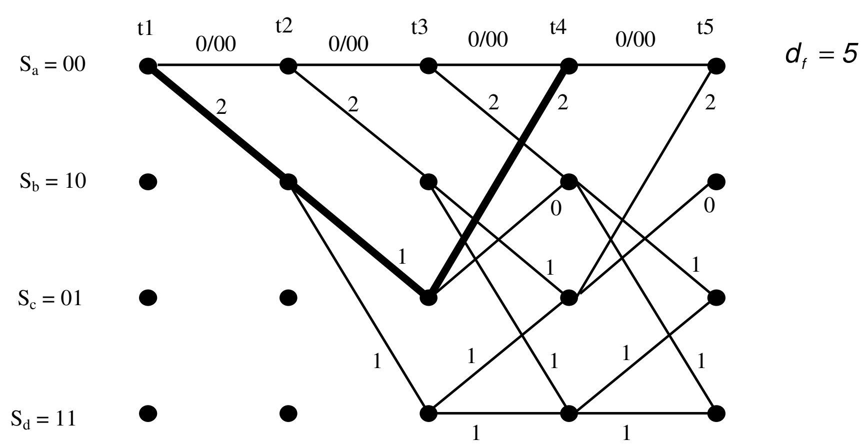
a) A binary convolutional error correcting code with $k = 1$, $n = 2$, $K = 2$, $g^{(1)}(D) = 1 + D + D^2$, and $g^{(2)}(D) = 1 + D^2$ has an encoder as the one seen in the following figure:



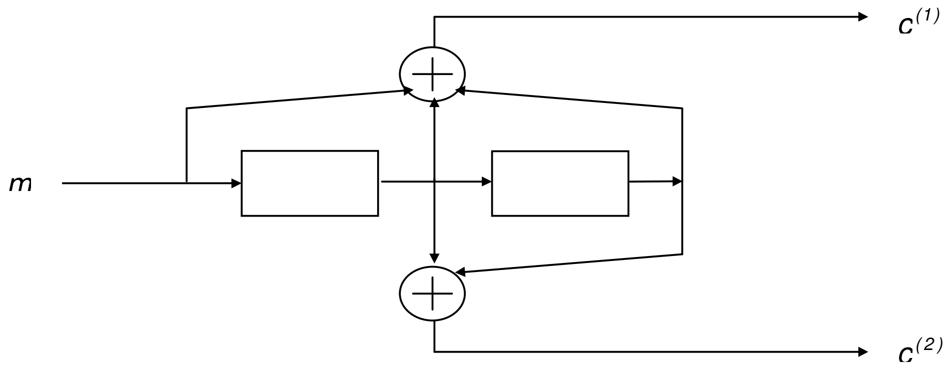
The code is non systematic, and of rate ½.
The following figure shows the corresponding trellis:



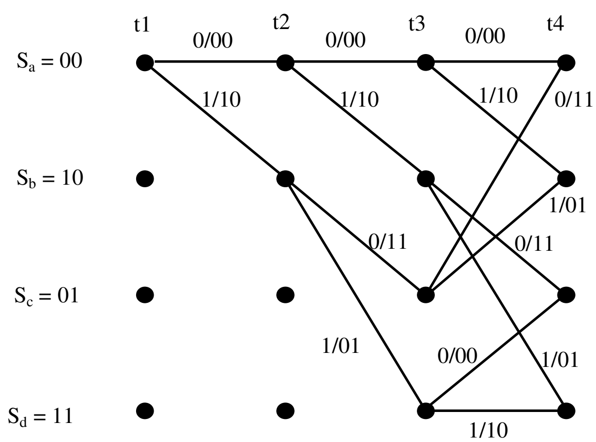
And the minimum Hamming distance is equal to $d_f = 5$:



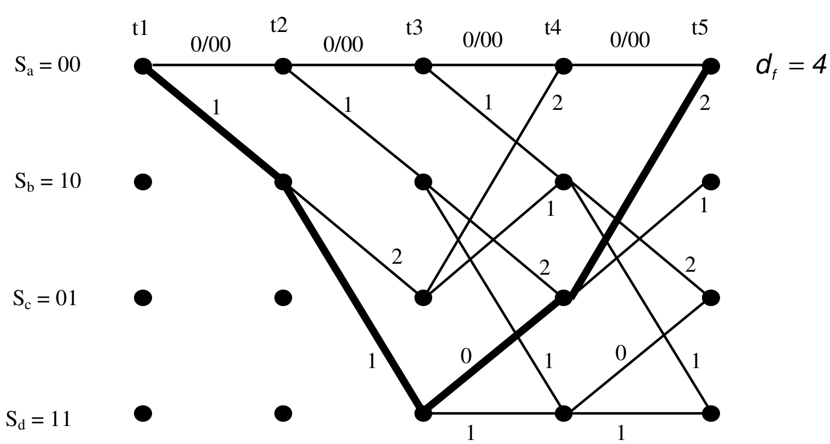
b) A binary convolutional error correcting code has $k=1$, $n=2$, $K=2$, $g^{(1)}(D)=1+D+D^2$, and $g^{(2)}(D)=D+D^2$. The corresponding convolutional encoder scheme is seen in the following figure:



The trellis is:



And the minimum Hamming distance is equal to $d_f = 4$:

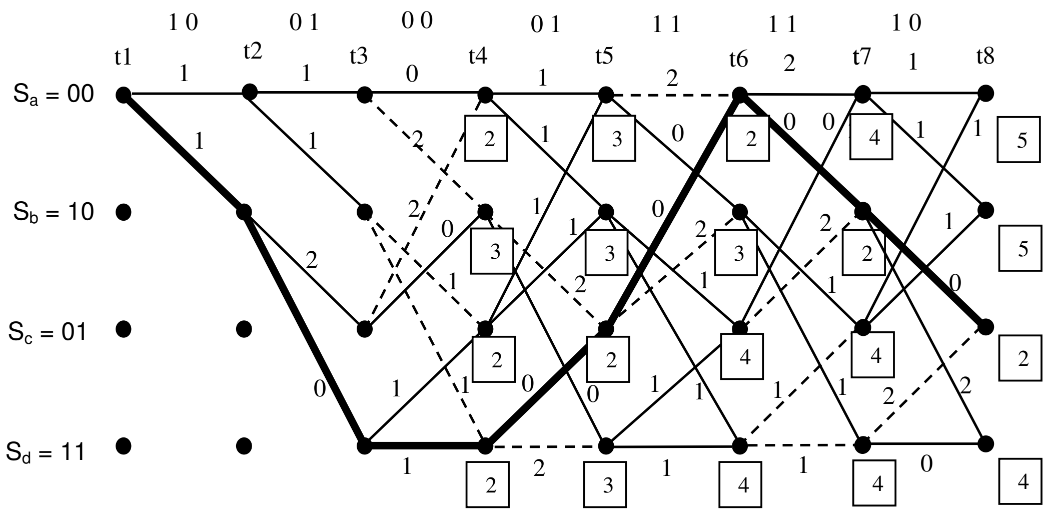


As it is seen, this code is the same as the one proposed in problem 6.5). The mistake is in the statement of problem 6.9), a typing error in $g^{(2)}(D) = D + D^2$, which should be $g^{(2)}(D) = 1 + D^2$, in order to agree with the solutions given in the book.

6.10) This problem suffers from the same typing mistake produced in problem 6.9). We will however give the detailed solutions for both cases.

a) Decoding of the sequence $\boldsymbol{s_r} = (10\ \ 01\,00\,01\ \ 11\ \ 11\ \ 10)$ when the binary convolutional error correcting code has $k = 1$, $n = 2$, $K = 2$, $g^{(1)}(D) = 1 + D + D^2$, and $g^{(2)}(D) = 1 + D^2$.

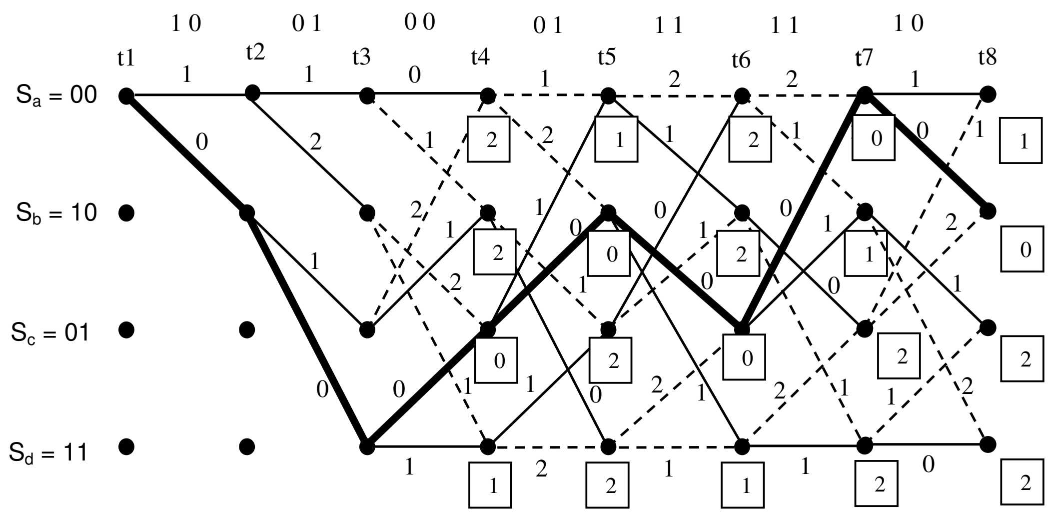
The decoding at the different time instants is shown in the following figure and the accumulative Hamming distances are determined in each step:



The decoded sequence is then $\boldsymbol{d} = (11\ \ 01\,10\,01\ \ 11\ \ 11\ \ 10)$, and the decoded message is $m = (1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 1\ \ 0)$. This is the answer given in the book, which as you can see, corresponds to the case $g^{(2)}(D) = 1 + D^2$. The proposed received sequence contained errors, which were corrected.

b) Decoding of the sequence $\boldsymbol{s_r} = (10\ \ 01\,00\,01\ \ 11\ \ 11\ \ 10)$ when the binary convolutional error correcting code has $k = 1$, $n = 2$, $K = 2$, $g^{(1)}(D) = 1 + D + D^2$, and $g^{(2)}(D) = D + D^2$.

The trellis is now different with respect to the case a). The decoding at the different time instants is shown in the following figure and the accumulative minimum Hamming distances are determined in each step:



The decoded sequence is then $\mathbf{d} = (10\ \ 01\, 00\, 01\ \ 11\ \ 11\ \ 10)$, that is, it is the received sequence, and the decoded message is $m = (1\ \ 1\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1)$. In other words, the received sequence is a code sequence.

This answer is not given in the book, which as you can see, corresponds to the case $g^{(2)}(D) = D + D^2$.

We apologize about this mistake that could lead you to some confusion.