2.1)

$$\mathbf{P} = [1 \quad 1], \qquad \mathbf{G} = [1 \quad 1 \quad 1], \qquad \mathbf{H} = [\mathbf{I}_{n-k} \quad \mathbf{P}^T] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad \mathbf{H}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Code table

m	**c**
0	*000*
1	*111*

Syndrome table

r	**S**
000	*00*
001	*11*
010	*01*
011	*10*
100	*10*
101	*01*
110	*11*
111	*00*

There are 8 error patterns, and the most likely ones are:

$\mathbf{e}$	$\mathbf{e} \circ \mathbf{H}^T$
000	*00*
001	*11*
010	*01*
100	*10*

Syndromes associated to the most likely error patterns, that are error patterns of one error, are all distinct, so that the code is able to correct error patterns of one bit ($t = 1$). It can also detect any error pattern of two bits ($l = 2$).

$\mathbf{r}$	$\mathbf{r} \circ \mathbf{H}^T$	$\mathbf{e}$	$\mathbf{r} \oplus \mathbf{e}$	$\hat{\mathbf{m}}$
000	*00*	*000*	*000*	*0*
001	*11*	*001*	*000*	*0*
010	*01*	*010*	*000*	*0*
011	*10*	*100*	*111*	*1*
100	*10*	*100*	*000*	*0*
101	*01*	*010*	*111*	*1*
110	*11*	*001*	*111*	*1*
111	*00*	*111*	*111*	*1*

2.2)

$d_{min} = 11$; $\qquad d_{min} \geq l + 1 \quad \Rightarrow \quad l = 10$

$\qquad d_{min} \geq 2t + 1 \quad \Rightarrow \quad t = 5$

2.3)

a) Since $l + t + 1 \geq d_{min}$, then $6 + 4 + 1 \geq d_{min} = 11$

$d_{min} = n - k + 1 = 11$, with $k_{min} = 1$, $n = 11$

b) It can detect n erasures and correct 10 erasures

c)
$d_{min} = n - k + 1 = 11$, with $k_{min} = 1$, $n = 11$

2.4)

0	0	0	0	0	0		0
0	1	1	1	0	0		3
1	0	1	0	1	0		3
1	1	0	1	1	0		4
1	1	0	0	0	1		3
1	0	1	1	0	1		4
0	1	1	0	1	1		4
0	0	0	1	1	1		3

a) $R_c = 3/6 = 1/2 = 0.5$

b) By selecting three linearly independent code vectors of the code, we can form the generator matrix. This matrix is formed so that the identity sub matrix is at its right side

$$\mathbf{G} = [\mathbf{P} \quad \mathbf{I}_k] = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{H}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

c) Since the code is a linear block code, the minimum Hamming distance of the code is the minimum weight seen in the above table, that is, $d_{min} = 3$

d) $d_{min} = 3$, $l = 2$, $t = 1$

e) Syndrome vector for the received vector $\mathbf{r} = (101011)$ is $S = r \circ H^T = (110)$, the error pattern-syndrome table is the following:

$\mathbf{e}$	$\mathbf{e} \circ \mathbf{H}^T$
100000	100
010000	010
001000	001
000100	011
000010	101
000001	110

Error is in the sixth bit, and then $\mathbf{e} = (000001)$, $\mathbf{c} = \mathbf{r} \oplus \mathbf{e} = (101010)$

2.5)

a) In a linear block code $C_b(5,2)$, $k = 2$, $n = 5$, there are $2^2 = 4$ codewords of length 5. Therefore the corresponding generator matrix is of the form:

$$\mathbf{G} = [\mathbf{P} \quad \mathbf{I}_k] = \begin{bmatrix} \mathbf{P} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{bmatrix}$$

The all zero codeword should belong to this code. From the possible 32 codewords of length 5, we can select $\mathbf{c_1} = (11110)$ and $\mathbf{c_2} = (10101)$ to form the generator matrix as:

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

This code has the following code table:

m	***c***	*w*
00	*00000*	0
01	*10101*	3
10	*11110*	4
11	*01011*	3

The minimum distance of the code is $d_{min} = w_{min} = 3$.
The code has a transpose parity check matrix of size $nx(n-k) = 5x3$, which means that the syndrome vectors are of size $1x3$. Then the number of non-zero syndrome vectors is 7. Since the code length is 5, we can not expect to have a code with an error correction capability of $t = 2$, because we would need to have at least:

$$\binom{n}{1} + \binom{n}{2} - 1 = \binom{5}{1} + \binom{5}{2} - 1 = 5 + 10 - 1 = 14$$

different syndrome vectors for such error correction capability.

b) The transpose parity check matrix of this code is:

$$H^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

2.6) A binary linear block code has the following generator matrix in systematic form:
2.7)

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

a) The parity check matrix $\mathbf{H}$ is:

$$H = \left[I_{n-k} \quad P^T \right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Parity check equations are of the form:

$c_0 \oplus m_0 \oplus m_1 \oplus m_2 = 0$ $\quad$ $c_1 \oplus m_0 \oplus m_2 = 0$ $\quad$ $c_2 \oplus m_1 \oplus m_2 = 0$

$c_3 \oplus m_0 \oplus m_1 = 0$ $\quad$ $c_4 \oplus m_0 = 0$ $\quad$ $c_5 \oplus m_1 = 0$

$c_6 \oplus m_2 = 0$ $\quad$ $c_7 \oplus m_0 \oplus m_1 \oplus m_2 = 0$ $\quad$ $c_8 \oplus m_0 \oplus m_2 = 0$

$c_9 \oplus m_1 \oplus m_2 = 0$

The code table is:

c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	m_0	m_1	m_2	w
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	1	1	1	1	0	0	1	8
1	0	1	1	0	1	0	1	0	1	0	1	0	7
0	1	0	1	0	1	1	0	1	0	0	1	1	7
1	1	0	1	1	0	0	1	1	0	1	0	0	7
0	0	1	1	1	0	1	0	0	1	1	0	1	7
0	1	1	0	1	1	0	0	1	1	1	1	0	8
1	0	0	0	1	1	1	1	0	0	1	1	1	8

b) The minimum Hamming distance of the code is $d_{min} = w_{min} = 7$

2.8)

a) The transpose parity check matrix and parity check equations of the code are as follows:

$$H^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$c_0 \oplus m_0 = 0$ $\quad c_1 \oplus m_0 = 0$ $\quad c_2 \oplus m_1 = 0$

$c_3 \oplus m_1 = 0$ $\quad c_4 \oplus m_0 \oplus m_1 = 0$ $\quad c_5 \oplus m_0 \oplus m_1 = 0$

b) Code rate is $R_c = 2/8 = 1/4$

Code table is:

m_0	m_1		c_0	c_1	c_2	c_3	c_4	c_5	m_0	m_1	w
0	0		0	0	0	0	0	0	0	0	0
0	1		0	0	1	1	1	1	0	1	5
1	0		1	1	0	0	1	1	1	0	5
1	1		1	1	1	1	0	0	1	1	6

The minimum Hamming distance of the code is $d_{min} = w_{min} = 5$, $t = 2$

c) $p = 10^{-3}$,

$$P_{be} \approx \binom{n-1}{t} p^{t+1} = \binom{7}{2} \left(10^{-3}\right)^3 = 21x10^{-9}$$

2.9)

a) The parity check matrix of the code is:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

b)

$$H^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

c) For the received vector $\mathbf{r} = (011111001011011)$

$$\mathbf{r} \circ \mathbf{H}^T = (0100) \oplus (0010) \oplus (0001) \oplus (0011) \oplus (0101) \oplus (1010) \oplus (0111) \oplus$$
$$(1110) \oplus (1011) \oplus (1111) = (0110)$$

This is the 8th row of $\mathbf{H}^T$, so that the error is at the 8th position

2.10)

a) The code is a shortened version of the Hamming code $C_b(15,11)$ that has as a parity submatrix:

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$2^{n-k} - 1 \geq n \qquad 2^{n-k} \geq 9 \qquad \Rightarrow \; n - k = 4 \qquad \Rightarrow \; k = 6$$

b) $n - k = 4$

c) the rate of the code is $R_c = 6/10 = 3/5$

d) The transpose of the parity check matrix is constructed using a submatrix $\mathbf{P}$ whose rows have at least two ones:

$$\mathbf{H}^T = \begin{bmatrix} \mathbf{I}_{n-k} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The generator matrix is then:

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

e) $\mathbf{c} = (111111) \circ \mathbf{G} = (0010111111)$

f) An error at the seventh digit of a code vector corresponds to an error pattern $\mathbf{e} = (0000001000)$, so that by inspection of the transpose parity check matrix, its seventh row is (0110), which is the corresponding syndrome.

2.11) $p = 10^{-3}$ in a BSC, block code with $n = 15$, $k = 11$, $d_{min} = 3$, $R_c = k / n = 11/15 = 0.733$

a) (FEC) mode

$$P_{we} \approx \binom{n}{t+1} p^{t+1} = \binom{15}{2} \left(10^{-3}\right)^2 = 1.04x10^{-4}$$

b) (ARQ) mode

$$R_c = k / n(1-p)^n = 11/15(1-0-001)^{15} = 0.722$$

$$P_{we} \approx \binom{n}{l+1} p^{l+1} = \binom{15}{3} \left(10^{-3}\right)^3 = 4.05x10^{-7}$$

2.12) FEC scheme, $P_{be} \le 10^{-4}$

a) 1) $t = 1$, $k = 26$, $n = 31$, $P_{be} = 10^{-4}$

$$P_{be} \approx \binom{n-1}{t} p^{t+1} = \binom{30}{1} p^{t+1} = 10^{-4}$$

$$p = Q\left(\sqrt{2R_c\left(E_b / N_0\right)}\right)$$

$$\frac{30!}{1!\,x29!} p^2 = 10^{-4}, \quad \Rightarrow \quad p^2 = \frac{10^{-4}}{30} = 3.33x10^{-6}$$

$p = 1.8x10^{-3}$, $\Rightarrow$ *using the function* $Q(k) \Rightarrow k = 2.9$

then

$$\sqrt{2\frac{26}{31}\left(E_b / N_0\right)} = 2.9, \quad \Rightarrow \quad E_b / N_0 = \frac{(2.9)^2 x31}{2x26} = 5.01 \quad , \quad \approx 7dB$$

2) $t = 2$, $k = 21$, $n = 31$, $P_{be} = 10^{-4}$

$$P_{be} \approx \binom{n-1}{t} p^{t+1} = \binom{30}{2} p^3 = 10^{-4}$$

$$p = Q\left(\sqrt{2R_c\left(E_b / N_0\right)}\right)$$

$$\frac{30!}{2!\,x28!} p^3 = 10^{-4}, \quad \frac{30x29}{2} p^3 = 10^{-4} \Rightarrow \quad p^3 = \frac{2x10^{-4}}{30x29} = 2.3x10^{-7}$$

$p = 6.12x10^{-3}$, $\Rightarrow$ *using the function* $Q(k) \Rightarrow k \approx 2.5$

then

$$\sqrt{2\frac{21}{31}\left(E_b / N_0\right)} = 2.5, \quad \Rightarrow \quad E_b / N_0 = \frac{(2.5)^2 x31}{2x21} = 4.61 \quad , \quad \approx 6.63dB$$

3) $t = 3$, $k = 16$, $n = 31$, $P_{be} = 10^{-4}$

$$P_{be} \approx \binom{n-1}{t} p^{t+1} = \binom{30}{3} p^4 = 10^{-4}$$

$$p = Q\left(\sqrt{2R_c\left(E_b / N_0\right)}\right)$$

$$\frac{30!}{3!\,x27!} p^4 = 10^{-4}, \quad \frac{30x29x28}{6} p^4 = 10^{-4} \Rightarrow \quad p^4 = \frac{6x10^{-4}}{30x29x28} = 2.46x10^{-8}$$

$p = 1.25\,x10^{-2}$, $\Rightarrow$ *using the function* $Q(k) \Rightarrow k \approx 2.15$

then

$$\sqrt{2\frac{16}{31}\left(E_b / N_0\right)} = 2.15, \quad \Rightarrow \quad E_b / N_0 = \frac{(2.15)^2 x31}{2x16} = 4.47 \quad , \quad \approx 6.5dB$$

The accuracy of this result is low, because it depends on the approximation done on the graphic of Q(k). We can say that options 2 and 3 are quite close to each other.

b) For uncoded transmission:

$$Q\left(\sqrt{2E_b / N_0}\right) = 10^{-4}, \quad \Rightarrow \quad E_b / N_0 = (3.7)^2 / 2 = 6.845, \quad 8.35dB$$

$$Gain = 8.35dB - 6.5dB \approx 1.85dB$$

2.13)

a) The transmission rate is the bit rate increased by the inverse of the rate of the code utilized. The increased transmission rates are $(12/11)r_b$, $(15/11)r_b$ and $(16/11)r_b$ respectively.

b) The required value of E_b / N_0 are:

1) $l = 1$, $k = 11$, $n = 12$, $P_{be} = 10^{-5}$

$$P_{be} \approx \binom{n-1}{l} p^{l+1} = \binom{11}{1} p^2 = 10^{-5}$$

$$p = Q\left(\sqrt{2R_c(E_b / N_0)}\right)$$

$$\frac{11!}{1!\,x10!} p^2 = 10^{-5}, \quad \Rightarrow \quad p^2 = \frac{10^{-5}}{11} = 9.1x10^{-7}$$

$p = 9.53x10^{-4}, \quad \Rightarrow$ *using the function* $Q(k) \Rightarrow \; k = 3.1$

then

$$\sqrt{2\frac{11}{12}(E_b / N_0)} = 3.1, \quad \Rightarrow \quad E_b / N_0 = \frac{(3.1)^2 x12}{2x11} = 5.24 \quad , \quad \approx 7.2dB$$

2) $l = 2$, $k = 11$, $n = 15$, $P_{be} = 10^{-5}$

$$P_{be} \approx \binom{n-1}{l} p^{l+1} = \binom{14}{2} p^3 = 10^{-5}$$

$$p = Q\left(\sqrt{2R_c(E_b / N_0)}\right)$$

$$\frac{14!}{2!\,x12!} p^3 = 10^{-5}, \; \frac{14x13}{2} p^3 = 10^{-5} \Rightarrow \quad p^3 = \frac{2x10^{-5}}{14x13} = 1.09x10^{-7}$$

$p = 4.8x10^{-3}, \quad \Rightarrow$ *using the function* $Q(k) \Rightarrow \; k \approx 2.6$

then

$$\sqrt{2\frac{11}{15}\left(E_b / N_0\right)} = 2.6, \quad \Rightarrow \quad E_b / N_0 = \frac{(2.6)^2 x15}{2x11} = 4.6 \quad , \approx 6.63dB$$

1) $l = 3$, $k = 11$, $n = 16$, $P_{be} = 10^{-5}$

$$P_{be} \approx \binom{n-1}{l} p^{l+1} = \binom{15}{3} p^4 = 10^{-5}$$

$$p = Q\left(\sqrt{2R_c\left(E_b / N_0\right)}\right)$$

$$\frac{15!}{3!\,x12!} p^4 = 10^{-5}, \quad \Rightarrow \quad p^4 = \frac{6x10^{-5}}{15x14x13} = 2.19x10^{-8}$$

$$p = 1.21x10^{-2}, \quad \Rightarrow using\ the\ function\ Q(k) \Rightarrow \ k \approx 2.3$$

then

$$\sqrt{2\frac{11}{16}\left(E_b / N_0\right)} = 2.3, \quad \Rightarrow \quad E_b / N_0 = \frac{(2.3)^2 x16}{2x11} = 3.85 \quad , \approx 5.85dB$$

Here again accuracy of results depend on graphic approximations done over the curve Q(k)

2.14)

The knowledge of the parity equations allow us to determine the parity check submatrix $\boldsymbol{P}$, which in turn is part, in a systematic block code, of the generator matrix $\boldsymbol{G}$. The reminder information is the identity submatrix, so that the parity check equations constitute enough information to construct the generator matrix of a linear block code in systematic form. Any non systematic version of the code has a generator matrix that is constructed by performing row operations using Gaussian elimination over the original systematic matrix.