Solutions to Problems of Chapter 1

1.1)

a)

In order to find the entropy we need to first calculate the self informations $I_i = log_2(1/P_i)$:

$$I_A = log_2(1/0.4) = 1.32$$
$$I_B = log_2(1/0.2) = 2.32$$
$$I_C\ log_2(1/0.1) = 3.32$$
$$I_E = I_F = log_2(1/0.05) = 4.32$$

where self informations are measured in bits.

Then the entropy of the source is the mean value of these self informations:

$$H(X) = \sum_{i=A}^{F} P_i I_i = 0.4x1.32 + 0.2x2.32 + 0.1x3.32 + 2x0.05x4.32 = 2.22 bits/symbol$$

b) The maximum possible source entropy is:

$$H_{max}(X) = \frac{1}{M}\sum log_2 M = log_2(M) = 2.58 bits/symbol$$

and the efficiency is:

$$eff = 2.22/2.58 = 86\%$$

The information emitted by this source can be compressed

1.2) Entropy for this case is calculated as usual as:

$$H(X) = \sum_{i=A}^{E} P_i I_i = (1/2)\,log_2(2) + (1/4)\,log_2(4) + (1/8)\,log_2(8) + 2x(1/16)\,log_2(16) = 1.875\ bits/symbol$$

the information contained by the sequence DADED is:

$$I_{seq} = I_A + 3I_D + I_E = log_2(2) + 3\,log_2(16) + log_2(16) = 17\ bits$$

1.3) The source entropy is calculated as:

$$H(X) = 0.2x\,log_2(1/0.2) + 0.8x\,log_2(1/0.8) = 0.7219\ bits/symbol$$

The transinformation of this channel can be determined using:

$$I(X,Y) = \Omega(\alpha + p - 2\alpha p) - \Omega(p)$$

where $\Omega(p) = p\,log_2(1/p) + (1-p)\,log_2\left(\frac{1}{(1-p)}\right)$

In this case:

$$\Omega(0.25) = 0.25\, log_2(1/0.25) + (0.75)\, log_2\left(\frac{1}{0.75}\right) = 0.8112$$

$$\alpha + p - 2\alpha p = 0.2 + 0.25 - 2x0.2x0.25 = 0.35$$

and

$$\Omega(0.35) = 0.35\, log_2(1/0.35) + (0.65)\, log_2\left(\frac{1}{0.65}\right) = 0.934$$

Therefore:

$$I(X,Y) = 0.934 - 0.8112 = 0.123\ bits/symbol$$

Capacity is the maximum value of $I(X,Y)$

$$C_s = max_{P(x_i)}\, I(X;Y) = 1 - \Omega(p) = 1 - 0.8112 = 0.189\ bits/symbol$$

1.4) For the BSC, the entropy is maximum when symbols are equally likely:

In $I(X,Y) = \Omega(\alpha + p - 2\alpha p) - \Omega(p)$, $\Omega(p)$ is constant if the channel is fixed, then $\Omega(\alpha + p - 2\alpha p)$ is maximum if $\alpha + p - 2\alpha p = 1/2$, thus,

$$\alpha(1 - 2p) + p = 1/2$$

$$\alpha = \frac{1/2 - p}{1 - 2p} = 1/2\,\frac{1 - 2p}{1 - 2p} = 1/2$$

The entropy $H(X)$ is maximum when the symbols are equally likely

$$H(X) = \Omega(p) = p\, log_2 \frac{1}{p} + (1 - p)\, log_2 \frac{1}{1 - p}$$

This function is depicted in chapter 1, and it is seen that its maximum value is verified when

$$p = 1 - p,$$

$$2p = 1$$

$$p = 1/2$$

This can be also determined obtaining the derivative of $H(X)$ with respect to p:

$$H(X) = \Omega(p) = p\, log_2 \frac{1}{p} + (1 - p)\, log_2 \frac{1}{1 - p} = p\, log_2 \frac{1}{p} + log_2 \frac{1}{1 - p} - p\, log_2 \frac{1}{1 - p} =$$

$$p[log_2(1) - log_2(p)] + log_2(1) - log_2(1 - p) - p[log_2(1) - log_2(1 - p)] =$$

$$-p\, log_2(p) - log_2(1 - p) + p\, log_2(1 - p)$$

so therefore

$$H'(X) = -p.\frac{1}{p} - log_2(p) + \frac{1}{1-p} + log_2(1-p) - p.\frac{1}{1-p} =$$

$$-1 - log_2(p) + \frac{1-p}{1-p} + log_2(1-p) = -log_2(p) + log_2(1-p) = 0$$

$$log_2 \frac{1-p}{p} = 0; \quad \frac{1-p}{p} = 1; \quad p = 1/2$$

1.5) Transinformation is calculated as follows:

$$I(X,Y) = \Omega(\alpha + p - 2\alpha p) - \Omega(p)$$

then $\alpha + p - 2\alpha p = 0.25 + 0.01 - 2x0.25x0.01 = 0.25 + 0.005 = 0.255$

$$\Omega(0.255) = 0.255x\, log_2 \frac{1}{0.255} + 0.745x\, log_2 \frac{1}{0.745} = 0.8191$$

on the other hand

$$\Omega(p) = \Omega(0.01) = 0.01x\, log_2(100) + 0.99x\, log_2\left(\frac{1}{0.99}\right) = 0.08079$$

therefore,

$$I(X,Y) = \Omega(\alpha + p - 2\alpha p) - \Omega(p) = 0.8191 - 0.0879 = 0.7383$$

Since $I(X,Y) = H(X) - H(X/Y)$

equivocation is equal to:

$$H(X/Y) = H(X) - I(X,Y)$$

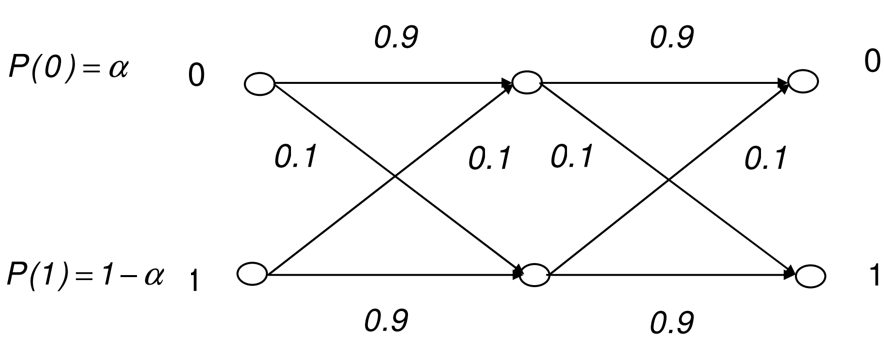
and we need to calculate $H(X)$:

$$H(X) = 0.25x\, log_2\left(\frac{1}{0.25}\right) + 0.75x\, log_2\left(\frac{1}{0.75}\right) = 0.8112$$

then,

$$H(X/Y) = H(X) - I(X,Y) = 0.8112 - 0.7383 = 0.073$$

1.6) Cascade of BSCs

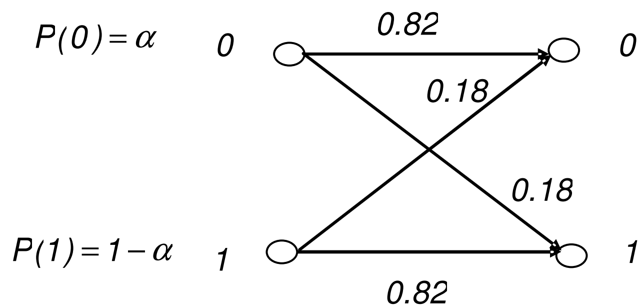


The composite channel can be understood as a single channel with probabilities:

$$0.9x0.9 + 0.1 + 0.1 = 0.82$$

and

$$2x0.9x0.1 = 0.18$$



$$C_s = max_{P(x_i)} I(X;Y) = 1 - \Omega(p) = 1 - [0.18x\, log_2\left(\frac{1}{0.18}\right) + 0.82x\, log_2\left(\frac{1}{0.82}\right) = 0.3199\ bits / symbol$$

1.7) For a binary channel over the binary alphabet and with transition probabilities:

$$\boldsymbol{P_{ch}} = \begin{bmatrix} 3/5 & 2/5 \\ 1/5 & 4/5 \end{bmatrix}$$

the a posteriori entropies are calculated as follows:

$$P(Y=0) = P(Y=0/X=0)P(X=0) + P(Y=0/X=1)P(X=1) = \frac{3}{5}\frac{1}{2} + \frac{1}{5}\frac{1}{2} = \frac{2}{5}$$

$$P(Y=1) = P(Y=1/X=0)P(X=0) + P(Y=1/X=1)P(X=1) = \frac{2}{5}\frac{1}{2} + \frac{4}{5}\frac{1}{2} = \frac{3}{5}$$

$$P(X=0/Y=0) = \frac{P(Y=0/X=0)P(X=0)}{P(Y=0)} = \frac{(3/5)(1/2)}{(2/5)} = \frac{3}{4}$$

$$P(X=0/Y=1) = \frac{P(Y=1/X=0)P(X=0)}{P(Y=1)} = \frac{(2/5)(1/2)}{(3/5)} = \frac{1}{3}$$

$$P(X=1/Y=1) = \frac{P(Y=1/X=0)P(X=1)}{P(Y=1)} = \frac{(4/5)(1/2)}{(3/5)} = \frac{2}{3}$$

$$P(X=1/Y=0) = \frac{P(Y=0/X=1)P(X=1)}{P(Y=0)} = \frac{(1/5)(1/2)}{(2/5)} = \frac{1}{4}$$

Then,

$$H(X/0) = \frac{3}{4}\log_2\left(\frac{4}{3}\right) + \frac{1}{4}\log_2(4) = 0.8112\ bit$$

$$H(X/1) = \frac{1}{3}\log_2(3) + \frac{2}{3}\log_2\left(\frac{3}{2}\right) = 0.9182\ bit$$

and the a priori entropy is equal to:

$$H(X) = \frac{1}{2}x\log_2(2) + \frac{1}{2}x\log_2(2) = 1bit$$

1.8) We have:

$p(x_1) = \alpha = 0.25$ $\quad p(x_2) = 1-\alpha = 1-0.25 = 0.75$

$p(y_1/x_1) = 1-p = 0.531$ $\quad p(y_2/x_1) = p = 0.469$ $\quad p(y_3/x_1) = 0$

$p(y_1/x_2) = 0$ $\quad p(y_2/x_2) = p = 0.469$ $\quad p(y_3/x_2) = 1-p = 0.531$

$$P(Y=0) = P(Y=0/X=0)P(X=0) = \alpha(1-p)$$
$$P(Y=E) = P(Y=E/X=0)P(X=0) + P(Y=E/X=1)P(X=1) = \alpha p + (1-\alpha)p = p$$
$$P(Y=1) = P(Y=1/X=1)P(X=1) = (1-p)(1-\alpha)$$
$$P(X=0/Y=0) = \frac{P(Y=0/X=0)P(X=0)}{P(Y=0)} = \frac{(1-p)\alpha}{\alpha(1-p)} = 1$$
$$P(X=0/Y=E) = \frac{P(Y=E/X=0)P(X=0)}{P(Y=E)} = \frac{p\alpha}{p} = \alpha = 0.25$$
$$P(X=1/Y=E) = \frac{P(Y=E/X=0)P(X=1)}{P(Y=E)} = \frac{p(1-\alpha)}{p} = 1-\alpha = 0.75$$
$$P(X=1/Y=1) = \frac{P(Y=1/X=1)P(X=1)}{P(Y=1)} = \frac{(1-p)(1-\alpha)}{(1-\alpha)(1-p)} = 1$$

$$I(X,Y) = (1-p)\Omega(\alpha) = (1-p)[\alpha\log_2\frac{1}{\alpha} + (1-\alpha)\log_2\frac{1}{(1-\alpha)} = (1-0.469)x0.8112 = 0.4307$$

$$C_s = 1-p = 0.531$$

and the equivocation is equal to:

$$H(X/Y) = H(X) - I(X,Y) = 0.8112 - 0.4307 = 0.38$$

1.9) In this non-symmetric channel:

$$P(Y=0) = P(Y=0/X=0)P(X=0) = 0.9x0.3 = 0.27$$
$$P(Y=E) = P(Y=E/X=0)P(X=0) + P(Y=E/X=1)P(X=1) = 0.1x0.3 + 0.2x0.7 = 0.17$$
$$P(Y=1) = P(Y=1/X=1)P(X=1) = 0.8x0.7 = 0.56$$

$$H(Y) = P(Y=0)\log_2\frac{1}{P(Y=0)} + P(Y=E)\log_2\frac{1}{P(Y=E)} + P(Y=1)\log_2\frac{1}{P(Y=1)} =$$

$$0.27x\log_2\frac{1}{0.27} + 0.17x\log_2\frac{1}{0.17} + 0.56\log_2\frac{1}{0.56} = 1.4130$$

$$H(Y/X) = P(Y=0/X=0)P(X=0)\log_2\frac{1}{P(Y=0/X=0)} +$$

$$P(Y=E/X=0)P(X=0)\log_2\frac{1}{P(Y=E/X=0)} +$$

$$+P(Y=E/X=1)P(X=1)\log_2\frac{1}{P(Y=E/X=1)} +$$

$$P(Y=1/X=1)P(X=1)\log_2\frac{1}{P(Y=1/X=1)}$$

$$H(Y/X) = 0.9x0.3\log_2\frac{1}{0.9} + 0.1x0.3\log_2\frac{1}{0.1} + 0.2x0.7\log_2\frac{1}{0.2} + 0.8x0.7\log_2\frac{1}{0.8} = 0.6461$$

Then,

$$I(X,Y) = H(Y) - H(Y/X) = 1.413 - 0.6460 = 0.767$$

The non-symmetry of the channel makes it difficult to maximise $I(X,Y)$ analytically, but an iterative approximation is a practical solution. Recalculating $I(X,Y)$ for various values of the source probabilities gives the following table:

α	0.3	0.4	0.45	0.5	0.55
$I(X,Y)$	0.767	0.841	0.858	0.862	0.854

Plotting these values on a graph, the maximum of the curve is found to be at $\alpha \approx 0.48$.
Then:

$P(Y=0) = 0.432$ $\quad P(Y=E) = 0.152$ $\quad P(Y=1) = 0.416$
$H(Y) = 1.4626$, $\quad H(Y/X) = 0.6005$

and therefore: $C_s = 0.862$

1.10) We have:

$$I(X,Y) = H(Y) - H(Y/X)$$

$$H(Y) = P(Y=0)\log_2\frac{1}{P(Y=0)} + P(Y=1)\log_2\frac{1}{P(Y=1)}$$

$$P(Y=0) = P(Y=0/X=0)P(X=0) + P(Y=0/X=1)P(X=1) =$$

$$= (1-p)\alpha + q(1-\alpha) = (1-q-p)\alpha + q$$

$$P(Y=1)=1-P(Y=0)$$
$$H(Y)=P(Y=0)\log_2\frac{1}{P(Y=0)}+(1-P(Y=0))\log_2\frac{1}{1-P(Y=0)}=\Omega((1-p-q)\alpha+q)$$
$$H(Y/X)=\sum_{i,j}P(x_i,y_j)\log_2\frac{1}{P(y_j/x_i)}=P(X=0)\left[P(Y=1/X=0)\log_2\frac{1}{P(Y=1/X=0)}+\right.$$
$$\left.P(Y=0/X=0)\log_2\frac{1}{P(Y=0/X=0)}\right]+P(X=1)\left[P(Y=1/X=1)\log_2\frac{1}{P(Y=1/X=1)}+\right.$$
$$\left.P(Y=0/X=1)\log_2\frac{1}{P(Y=0/X=1)}\right]=$$

$$=\alpha\left[p\log_2(p)+(1-p)\log_2\frac{1}{(1-p)}\right]+(1-\alpha)\left[q\log_2(q)+(1-q)\log_2\frac{1}{(1-q)}\right]$$
$$I(X,Y)=H(Y)-H(Y/X)=\Omega(q+(1-p-q)\alpha)-\alpha\Omega(p)-(1-\alpha)\Omega(q)$$

1.11) We first calculate the following probabilities:

$$P(Y=0)=0.9x0.25+0.02x0.75=0.24$$
$$P(Y=E)=0.08x0.25+0.08x0.75=0.08$$
$$P(Y=1)=0.9x0.75+0.02x0.25=0.68$$

$$H(Y)=0.24\log_2\frac{1}{0.24}+0.08\log_2\frac{1}{0.08}+0.68\log_2\frac{1}{0.68}=1.16399$$
$$H(Y/X)=0.9x0.25\log_2\frac{1}{0.9}+0.08x0.25\log_2\frac{1}{0.08}+0.02x0.25\log_2\frac{1}{0.02}+0.02x0.75\log_2\frac{1}{0.02}+$$
$$0.08x0.75\log_2\frac{1}{0.08}+0.9x0.75\log_2\frac{1}{0.9}=0.54118$$

$$I(X,Y)=1.16399-0.54118=0.6228$$

Note that since $H(Y/X)$ is independent of the source probabilities, then capacity will be achived when $\alpha=1/2$:

$$I(X,Y)=I_{max}(X,Y)=C_s$$

$$P(Y=0)=0.46$$
$$P(Y=E)=0.08$$
$$P(Y=1)=0.46$$

$$H(Y)=0.46\log_2\frac{1}{0.46}+0.08\log_2\frac{1}{0.08}+0.46\log_2\frac{1}{0.46}=1.322$$
$$H(Y/X)=0.54118$$

$$I(X,Y) = I_{max}(X,Y) = C_s = 0.781$$

$$\% = \frac{0.6228}{0.781} = 79.6\%$$

1.12) a) Capacity is calculated as:

$$C = B \log_2(1 + S/N) = 3000 \log_2(1 + 1000) = 29902 \; bps$$

b)

If $C = R = 19200 \; bps$

$$\frac{19200}{3000} = \log_2(1 + S/N) \quad \Rightarrow \quad S/N = 2^{\frac{19200}{3000}} - 1 = 83{,}44 \quad \Rightarrow 19.21 dB$$

1.13) Capacity is calculated as:

$$C = B \log_2(1 + S/N) = 25000 \log_2(1 + 63{,}095) = 150053 \; bps$$